pressible frictionless fluid round a solid having the form of the ring; for the components of velocity u, v, w, are subject to precisely the same conditions as are α, β, γ. If the density be unity, the kinetic energy T of the motion has the expression

$$\mathrm{T} = \frac{\mathrm{L}}{8\pi} \times (\text{circulation})^2, \tag{46}$$

L having the value given in (44).

P.S., March 4.—Sir W. D. Niven, who in 1881 verified some other results for self-induction—those numbered (11), (12) in the paper referred to—has been good enough to confirm the formulæ (1), (28) of the present communication, in which I differ from M. Wien.

The Diffusion and Mobility of Ions in a Magnetic Field.

By John S. Townsend, F.R.S., Wykeham Professor of Physics, Oxford.

(Received March 11,—Read April 25, 1912.)

1. When the motion of ions in a gas takes place in a magnetic field the rates of diffusion and the velocities due to an electric force may be determined by methods similar to those given in a previous paper.*

The effect of the magnetic field may be determined by considering the motion of each ion between collisions with molecules. The magnetic force causes the ions to be deflected in their free paths, and when no electric force is acting the paths are spirals, the axes being along the direction of the magnetic force. If H be the intensity of the magnetic field, e the charge, and m the mass of an ion, then the radius r of the spiral is $mv/\mathrm{H}e$, v being the velocity in the direction perpendicular to H. The distance that the ion travels in the interval between two collisions in a direction normal to the magnetic force is a chord of the circle of radius r. The average lengths of these chords may be reduced to any fraction of the projection of the mean free path in the direction of the magnetic force, so that the rate of diffusion of ions in the directions perpendicular to the magnetic force is less than the rate of diffusion in the direction of the force.

In the previous paper it has been shown that the coefficient of diffusion K is a measure of the rate of increase of the mean square R^2 of the distance of any distribution from any point, $d\mathrm{R}^2/dt$ being equal to 6K. Since

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* 'Roy. Soc. Proc.,' February, 1912, vol. 86.

$R^2 = x^2+y^2+z^2$ it follows that $2K = dx^2/dt$, and $4K = d\rho^2/dt$, where ρ^2 is the mean of the squares of the distances of the ions from the axis of z.

When a magnetic force H is applied in the direction of the axis of z, the rate of diffusion K_h in the directions perpendicular to z is less than the rate of diffusion K along z, which is unaltered by the magnetic field. As the phenomena are of special interest in connection with negative ions in the electronic state when magnetic fields produce large effects, the collisions with molecules may be supposed to affect the motion of the ions in the same way as the motion of small bodies is affected by colliding with comparatively large particles. All directions of motion of an ion after collision with molecules may therefore be supposed to be equally probable, so that the mean values of the velocities of the ions after colliding will be zero. Since the magnetic field only alters the direction of motion, the number of collisions that an ion makes per second is not affected, and the intervals between the collisions will be distributed as in the ordinary case, and will have the same mean value T. The number of intervals that exceed the time t out of a total number N will be $Ne^{-t/T}$.

In the following investigations two expressions occur which may be found in terms of the mean time between collisions. These are the series of cosines: $\cos \omega t_1 + \cos \omega t_2 + \text{etc.}$; and the series of sines: $\sin \omega t_1 + \sin \omega t_2 + \text{etc.}$, t_1, t_2, etc., being intervals between a large number N of consecutive collisions that an ion makes with molecules. The number of intervals that lie between the values t and $t+dt$ is $\frac{N}{T}e^{-t/T}dt$, so that the series of cosines may be expressed as the integral

$$\frac{N}{T}\int_0^\infty \cos \omega t\, e^{-t/T}dt,$$

This expression may be integrated by parts and its value is $N/(1+\omega^2T^2)$. Similarly, the series of sines will be found to be equal to $N\omega T/(1+\omega^2T^2)$.

2. The rate of increase of the mean square of the distance of any distribution from the axis of z, $d\rho^2/dt$, may be determined by considering the motion of an ion parallel to the plane of xy.

Let x and y be the ordinates of an ion at any time t after a collision. The equations of motion are—

$$m\frac{d^2x}{dt^2} = -He\frac{dy}{dt}, \qquad m\frac{d^2y}{dt^2} = He\frac{dx}{dt},$$

and the velocities $\dot{x}$ and $\dot{y}$ are given by equations of the form

$$m^2\frac{d^2\dot{x}}{dt^2} + H^2e^2\dot{x} = 0,$$

so that $\dot{x} = A\sin(\omega t+\alpha)$ and $\dot{y} = -A\cos(\omega t+\alpha)$.

Let $\dot{x}_1$, $\dot{y}_1$ be the velocities at the beginning of the first interval, and let the first collision occur in a time t_1; $\dot{x}_2$, $\dot{y}_2$ the velocities after the first collision, and t_2 the interval between the first and second collision, etc.

In the first interval, the distances δx_1 and δy_1 traversed by the ions are

$$\delta x_1 = \frac{A}{\omega}[\cos\alpha_1 - \cos(\omega t_1 + \alpha_1)] \quad \text{and} \quad \delta y_1 = \frac{A}{\omega}[\sin\alpha_1 - \sin(\omega t_1 + \alpha_1)].$$

Let x_0, y_0 be the initial position of the ion, then after the time $\mathrm{NT} = t_1 + t_2 + \text{etc.}$, the position will be

$$x_0 + \sum_{s=1}^{s=\mathrm{N}} \delta x_s \quad \text{and} \quad y_0 + \sum_{s=1}^{s=\mathrm{N}} \delta y_s,$$

and the square of the distance from the axis will have increased by the amount

$$2x_0 \Sigma\, \delta x_s + 2y_0 \Sigma\, \delta y_s + \Sigma(\delta x_s)^2 + \Sigma\,(\delta y_s)^2 + 2\Sigma\,(\delta x_r \delta x_s + \delta y_r \delta y_s).$$

The latter terms are zero, since, on the average, the sum of the cosines of the angles which any path makes with the consecutive paths vanishes.

Hence, for any ion, the square of the distance changes by the amount

$$2x_0 \Sigma\, \delta x_s + 2y_0 \Sigma\, \delta y_s + \Sigma(\delta x_s)^2 + \Sigma(\delta y_s)^2.$$

For a large number of ions starting from the point x_0, y_0, the average values of $\Sigma\, \delta x_s$ and $\Sigma\, \delta y_s$ vanish, since the ions are free to move in all directions. Hence the rate of change in the mean square of the distance of the distribution from the axis $d\rho^2/dt$ is $[\Sigma(\delta x_s)^2 + \Sigma(\delta y_s)^2]/\mathrm{NT}$.

Substituting for δx_s and δy_s their values, the quantity $d\rho^2/dt$ becomes $\dfrac{2}{\mathrm{NT}\omega^2}\Sigma\, \mathrm{A}_s^2(1 - \cos\omega t_s)$.

The velocities A_s are independent of the times t_s, so that the mean value of A^2 may be substituted for A_s^2 in this expression. Also, since A_s is the velocity in the plane xy, the mean value of A_s^2 is $\frac{2}{3}\mathrm{V}^2$, V being the mean velocity of agitation of the ions; and, since the series of cosines $\Sigma \cos\omega t_s$, is equal to $\mathrm{N}/(1 + \omega^2\mathrm{T}^2)$, the above expression reduces to

$$\frac{d\rho^2}{dt} = \frac{2\mathrm{N}}{\mathrm{NT}\omega^2} \times \frac{2\mathrm{V}^2}{3} \times \frac{\omega^2\mathrm{T}^2}{1 + \omega^2\mathrm{T}^2} = \frac{4\mathrm{V}^2\mathrm{T}}{3(1 + \omega^2\mathrm{T}^2)}.$$

The rate of diffusion K along the direction of the magnetic force is $\frac{1}{3}\lambda\mathrm{V}$ or $\frac{1}{3}\mathrm{V}^2\mathrm{T}$. Hence

$$\frac{d\rho^2}{dt} = \frac{4\mathrm{K}}{1 + \omega^2\mathrm{T}^2},$$

3. The motion of the ions may therefore be expressed in the usual form by the equations

$$\frac{nu}{\mathrm{K}_h} = -\frac{dn}{dx}, \quad \frac{nv}{\mathrm{K}_h} = -\frac{dn}{dy}, \quad \frac{nw}{\mathrm{K}} = -\frac{dn}{dz}.$$

The change of the number of ions n per cubic centimetre at any point is

$$-\frac{dn}{dt}=\frac{d(nu)}{dx}+\frac{d(nv)}{dy}+\frac{d(nw)}{dz}=\mathrm{K}_h\left(\frac{d^2n}{dx^2}+\frac{d^2n}{dy^2}\right)+\mathrm{K}\frac{d^2n}{dz^2}.$$

In this notation $d\rho^2/dt$ is given by the equation

$$\frac{d\rho^2}{dt}\iiint n\,dx\,dy\,dz=\iiint(x^2+y^2)\frac{dn}{dt}\,dx\,dy\,dz,$$

and it is easy to see that when the above value is substituted for dn/dt, the integral on the right of this equation reduces to $4\mathrm{K}_h\iiint n\,dx\,dy\,dz$.

Hence
$$\mathrm{K}_h=\frac{\mathrm{K}}{1+\omega^2\mathrm{T}^2}.$$

The equations of motion of ions in a magnetic field are usually given in the following form:—

$$\frac{1}{\mathrm{K}}(pu)=-\frac{dp}{dx}+ne\mathrm{H}v,\qquad \frac{1}{\mathrm{K}}(pv)=-\frac{dp}{dy}-ne\mathrm{H}u,$$

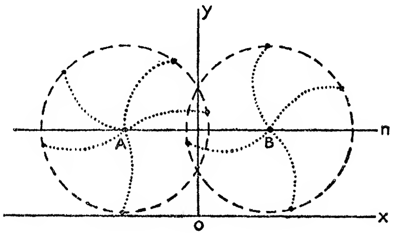
p being the partial pressure of the ions and H the magnetic force in the direction of the axis of z.

These equations are obtained on the supposition that the magnetic force does not affect the rate of diffusion K directly, but acts as if the charge per cubic centimetre ne were moving with velocities u and v. The latter supposition is legitimate for most purposes, since nu represents the excess of the number of ions crossing unit area of the plane $x =$ constant per second in one direction over the number crossing in the opposite direction. The validity of these assumptions when applied to the motion in a magnetic field may be tested by examining the conclusions to which the equations lead in a particular case. On eliminating v, the first equation becomes

$$\frac{1}{\mathrm{K}_h}(pu)=-\frac{dp}{dx}-\frac{\mathrm{H}e\mathrm{T}}{m}\cdot\frac{dp}{dy}.$$

So that when $dp/dx = 0$, there is a velocity parallel to the axis of x due to the slope of pressure dp/dy. Hence, if the surfaces of equal pressure be planes parallel to the plane $y = 0$, there would appear to be more ions crossing the plane $x = 0$ in one direction than in the other.

Let a number of ions begin their paths at a point A and collide with molecules of the gas after traversing a certain distance a from the point at which they started. The points at which they arrive will be situated symmetrically on a circle, and a certain proportion will cross the plane $x = 0$ and collide with molecules on the opposite side. From the point B, at which the number of ions per cubic centimetre is the same as at A, an equal number of ions will start and the same proportion of them will cross the plane $x = 0$ and collide with molecules on the side opposite to B. There will, therefore, be no tendency for the ions to move across the plane $x = 0$



and the only effect of the magnetic force is to cause the ions to move in circular paths, and so shorten their effective mean free paths along the direction of the axes of x and y. Hence $u = 0$ when $dn/dx = 0$. Apparently, therefore, the original equations which lead to an expression for pu involving dp/dy cannot be correct.

4. When ions are moving in an electric field and a magnetic force H is applied along the axis of z, the motion in that direction is unchanged, but in the directions x and y the velocities are altered. Taking the axis of x as the direction of the electric force X, the equations of motion become

$$m\frac{d^2x}{dt^2} = \mathrm{X}e - \mathrm{H}e\frac{dy}{dt}, \qquad m\frac{d^2y}{dt^2} = \mathrm{H}e\frac{dx}{dt},$$

so that the velocities $\dot{x}$ and $\dot{y}$ are given by the equations

$$\dot{x} = \mathrm{A}\sin(\omega t + \alpha) \quad \text{and} \quad \dot{y} = \frac{\mathrm{X}}{\mathrm{H}} - \mathrm{A}\cos(\omega t + \alpha),$$

the velocities at the beginning of the path being $\mathrm{A}\sin\alpha$ and $\mathrm{X}/\mathrm{H} - \mathrm{A}\cos\alpha$. Let $\dot{x}_1$, $\dot{x}_2$, etc., $\dot{y}_1$, $\dot{y}_2$, etc., be the velocities after a number of consecutive collisions of an ion with molecules of the gas.

The distances that an ion travels in the first interval t_1 are

$$\delta x_1 = \frac{\mathrm{A}_1}{\omega}[\cos\alpha_1 - \cos(\omega t_1 + \alpha_1)] \quad \text{and} \quad \delta y_1 = \frac{\mathrm{X}}{\mathrm{H}}t_1 + \frac{\mathrm{A}_1}{\omega}[\sin\alpha_1 - \sin(\omega t_1 + \alpha_1)].$$

The distances traversed in the time $\mathrm{NT} = t_1 + t_2 + \ldots + t_n$ are

$$\Sigma\,\delta x_s = \frac{\mathrm{NX}}{\mathrm{H}\omega} - \frac{1}{\omega}\Sigma\left(\frac{\mathrm{X}}{\mathrm{H}} - \dot{y}_s\right)\cos\omega t_s + \frac{1}{\omega}\Sigma\,\dot{x}_s\sin\omega t_s,$$

$$\Sigma\,\delta y_s = \frac{\mathrm{NXT}}{\mathrm{H}} - \frac{1}{\omega}\Sigma\left(\frac{\mathrm{X}}{\mathrm{H}} - \dot{y}_s\right)\sin\omega t_s - \frac{1}{\omega}\Sigma\,\dot{x}_s\cos\omega t_s.$$

The terms independent of X obviously vanish in estimating the mean displacements of a group of ions starting from the origin, and the velocities U_h and V_h along the axes become

$$U_h = \frac{\Sigma\,\delta x_s}{NT} = \frac{X}{H\omega T} - \frac{X}{H\omega NT}\cdot\frac{N}{1+\omega^2T^2} = \frac{X}{H\omega T}\cdot\frac{\omega^2T^2}{1+\omega^2T^2},$$

$$V_h = \frac{\Sigma\,\delta y_s}{NT} = \frac{X}{H} - \frac{X}{H\omega NT}\cdot\frac{N\omega T}{1+\omega^2T^2} = \frac{X}{H}\cdot\frac{\omega^2T^2}{1+\omega^2T^2};$$

and, since $\omega = He/m$, the expressions for the velocities reduce to

$$U_h = \frac{Xe}{m}\cdot\frac{T}{1+\omega^2T^2} \quad\text{and}\quad V_h = \frac{Xe}{m}\cdot\frac{\omega T^2}{1+\omega^2T^2}.$$

The velocity U when the magnetic force is zero is $\frac{Xe}{m}T$. Hence

$$U_h = \frac{U}{1+\omega^2T^2} \quad\text{and}\quad V_h = U_h\frac{He}{m}T.$$

Also the following relation holds, $U_h/K_h = U/K$, which shows that the velocity in the direction of the electric force is reduced by the magnetic field in the same proportion as the rate of diffusion.

5. The equations of motion from which the number of ions per cubic centimetre at any point in the gas may be determined when electric and magnetic forces are acting are

$$\frac{pu}{K_h} = -\frac{dp}{dx}+nXe, \qquad \frac{pv}{K_h} = -\frac{dp}{dy}+nXe\frac{He}{m}T, \qquad \frac{pw}{K} = -\frac{dp}{dz}+nZe,$$

the component of the electric force along the axis of y being zero.

6. When a stream of ions is moving under the action of an electric force the deflection of the stream from the direction of the electric force may be measured by the ratio

$$\frac{V_h}{U_h} = \frac{He}{m}T = \tan\theta.$$

The deflection varies with the pressure, since T the mean interval between collisions is inversely proportional to the pressure of the gas. Also when e/m is constant the deflection is independent of the electric force.

The value of $\tan\theta$ may be written in various forms, thus

$$\tan\theta = \frac{HU_h}{X}\left(1+\frac{H^2e^2T^2}{m^2}\right);$$

so that when the deflection is small $\tan\theta = HU_h/X$.

In this case the free paths are only slightly curved by the action of the magnetic force, for the radius r of the circle described by the ion in the plane $z = 0$ is mV/He, and the mean length s of the arc of the circle described between collisions is vT, hence

$$s/r = HeT/m = \tan\theta.$$

The length of the arc s described by the ion is therefore small compared with the radius of the circle.

When θ is large the velocity in the direction of the force becomes

$$U_h = \frac{Xe}{m} \cdot \frac{1}{\omega^2 T} = \frac{Xm}{H^2 e T}, \quad \text{so that} \quad \tan\theta = \frac{HeT}{m} = \frac{X}{HU_h}.$$

In this case the arc of the circle described between two collisions is large compared with the radius.

The velocities of ions U under an electric force in gases at low pressures may thus be easily determined by producing a small deflection θ of a stream with a magnetic force, since $U = U_h$ when θ is small. Also the theory may be tested by observing the effect of a magnetic force on the diffusion of a narrow stream moving under an electric force. In this case when the magnetic force coincides with the electric force, the motion arising from diffusion in directions normal to the force is reduced in the proportion K_h/K, so that the ions are kept together in a narrower stream.

There are several well known phenomena connected with the magnetic rays that occur in discharge tubes, in which remarkable effects are obtained by magnetic forces.

The observed effects are qualitatively in accordance with the above theory, but the conditions under which they are obtained and the system of forces that is called into play are so complicated that an accurate comparison with the theory would not be possible.

A New Treatment of Optical Aberration.

By R. A. Sampson, F.R.S.

(Received March 15,—Read May 23, 1912.)

(Abstract.)

A method is developed by which Gauss's method of relating original and emergent rays in a coaxial optical system

$$\left.\begin{aligned} y &= \beta x + b, \\ z &= \gamma x + c, \end{aligned}\right\} \qquad \left.\begin{aligned} y' &= \beta' x' + b', \\ z' &= \gamma' x' + c', \end{aligned}\right\}$$

by means of a transformation,

$$b' = Gb + H\beta, \qquad \beta' = Kb + L\beta, \qquad c' = Gc + H\gamma, \qquad \gamma' = Kc + L\gamma,$$

where $$GL - HK = \mu/\mu' = N,$$

may be applied so as to include the aberrations of the third order.